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PSUEDO CUTS AND THEIR APPLICATION

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INTRODUCTION

Consider a non-oriented planar graph G and the dual graph [1,2,3] G' of G. It is known that a circuit of G is a cut set of G'. Here we will introduce a set called a "psuedo cut" so that a special psuedo cut in G' is a path in G. When a linear graph is non-planar, a psuedo cut will have no relationship with a path in a linear graph. However, there is a similarity between the properties of paths and psuedo cuts. Let C be a class of all possible circuits, edge disjoint unions of circuits in a linear graph G and the empty set $\emptyset$. Let p be a path between vertices t and u in G. We define $C\Phi_{tu}$ as

$$C\Phi_{tu} = \{x : x = p \oplus c_r,\ c_r \in C\}$$

Then all possible paths between t and u in G are in $C\Phi_{tu}$ [4]. Also, it is known that $(C\Phi_{tu} \cup C)$ is a group under $\oplus$. Here we will show the similar relationship as follows: Let S be the class of all possible cut sets, edge disjoint unions of cut sets in a linear graph G and $\emptyset$. Let d be a psuedo cut. Now we define SD(d) as

$$SD(d) = \{x : x = d \oplus s_r,\ s_r \in S\}$$

Then $\{SD(d) \cup S\}$ is a group under $\oplus$.

The second part shows the relationship among classes of psuedo cuts which leads to an important property that the collection of all possible classes of psuedo cuts of a nonseparable linear graph is an Abelian group under the cross ring sum $\otimes$ operation. This property is exactly the same as that of the collection of the classes of paths [5].

The last part shows that a psuedo cut is a subset of a set of chords in a linear graph. Then we apply this property to obtain all possible sets of chords in a nonseparable graph.

PRELIMINARY

Definition 1: Class S is the collection of all possible cut sets, all possible edge disjoint unions of cut sets and the empty set $\emptyset$ in a linear graph. Class S_s is a subclass of S which consists of all possible cut sets and $\emptyset$.

Definition 2: Let U be a class of sets. Then, "Min U" is a subclass of U such that (1) if $\emptyset$ is in U, $\emptyset$ is in Min U, (2) for any set $u_1 \neq \emptyset$ in (U - Min U)*, there exists a set $u_2 \neq \emptyset$ in Min U such that $u_1 \supset u_2$, and (3) for any two sets $u_r \neq \emptyset$ and $u_t \neq \emptyset$ in Min U, $u_r \not\supset u_t$.

Example 1: Suppose sets $u_1 = \{abc\}$ and $u_2 = \{a\}$ are in U. Then u_1 is not in Min U because $u_2 \subset u_1$.

From the above definitions, it can be seen that

$$S_s = \text{Min } S \tag{1}$$

Definition 3: Let U and V be classes of sets. Then $U \otimes V$ is defined as

$$U \otimes V = \text{Min } \{x: x = u \oplus v,\ u \in U,\ v \in V\} \tag{2}$$

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*A-B is a class of sets each of which is in A but not in B.

where $u \oplus v = (u-v) \cup (v-u)$. For convenience, $\otimes$ is called a cross ring sum. When there is only one set in each of U and V, $\otimes$ becomes $\oplus$.**

PSUEDO CUT IN CLASS D(e)

Definition 4: Let G be a linear graph and edge e is in G. Then class D(e) is defined as

$$D(e) = \text{Min}\ \{x : x = \{e\} \oplus s,\ s \in S\} \tag{3}$$

under the condition that set $\{e\}$ is not a cut set in G.

D(e) is called a class of psuedo cuts of G with respect to edge e, and each set in D(e) is called a psuedo cut.

Since $\emptyset$ is in S, set $\{e\}$ is in D(e). Thus edge e is a psuedo cut. It is also clear that every psuedo cut in D(e) except $\{e\}$ becomes a cut set by adding edge e.

Example 2: Class S_s of a linear graph G in Fig. 1 consists of the following sets:

$$\emptyset, \{e_0e_1e_3\}, \{e_0e_1e_4\}, \{e_0e_2e_3\}, \{e_0e_2e_4\}, \{e_1e_2\} \text{ and } \{e_3e_4\}$$

Class S of G consists of all sets in S_s and $\{e_1e_2e_3e_4\}$.

The class $D(e_0)$ which can be obtained by Equation (3) or directly from G, consists of $\{e_0\}$, $\{e_1e_3\}$, $\{e_1e_4\}$, $\{e_2e_3\}$, and $\{e_2e_4\}$.

**Ring sum $\oplus$ of x and y is defined as $x \oplus y \subset x\bar{y} \cup \bar{x}y = (x-y) \cup (y-x)$.

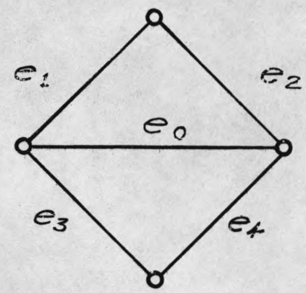

Fig. 1. A Linear Graph G

It is known that the ring sum of two paths both of which are between two vertices p and q in a non-oriented linear graph is either a circuit or an edge disjoint union of circuits. Two psuedo cuts with respect to edge e have the similar property which is given by the following theorem.

Theorem 1: Let d_1 and d_2 be psuedo cuts in D(e). Then $d_1 \oplus d_2$ is a set in S.

Proof: Since $d_i \cup \{e\}$ (i=1,2) is a cut set of G by Equation (3), we have

$$d_1 \oplus d_2 = (d_1 \oplus \{e\}) \oplus (d_2 \oplus \{e\}) = s_1 \oplus s_2 \qquad (4)$$

Notice if $d_i \neq \{e\}$, $d_i \oplus e = d_i \cup \{e\}$ which is a cut set s_i in S. If $d_i = \{e\}$, then $d_i \oplus \{e\} = \emptyset$ which is also in S. It is obvious that $s_1 \oplus s_2$ is in S. Q.E.D.

Lemma 1: Let d be a set in D(e) and s be a set in S. Then $d \oplus s$ is either a psuedo cut in D(e) or an edge disjoint union of a psuedo cut in D(e) and a set in S.

The proof is similar to that of Theorem 1.

In order to obtain a class which contains psuedo cuts and sets, each of which is an edge disjoint union of a psuedo cut and a set in S, we define the following:

Definition 5: For a given G and S of G, class SD(e) is defined as

$$SD(e) = \{x : x = \{e\} \oplus s,\ s \in S\} \tag{5}$$

The difference between D(e) and SD(e) is that there is no (Min.) in Equation (5). Thus, SD(e) is the collection of all possible psuedo cuts in D(e) and all possible sets each of which is an edge disjoint union of a psuedo cut in D(e) and cut sets in S.

From Theorem 1 and Lemma 1, we have the following:

Lemma 2: Let $SD(e) \cup S$ be the class of all sets in SD(e) and S of G. Then $SD(e) \cup S$ is an Abelian group under the operation $\oplus$.

It can be seen by Lemma 2 that the relationship among psuedo cuts in D(e) are the same as that of paths between two vertices and circuits in a linear graph.

PROPERTIES OF CLASS D(e) OF PSUEDO CUTS

By Definition 1, class S_s is a subset of class S. Notice that S is a class of cut sets, edge disjoint unions of cut sets, and the empty set. In order to obtain class D(e) (Definition 4) by Equation (3) we must know class S of a linear graph G. Because of "Min" in Equation (3), we can replace S in Equation (3) by S_s which is given by the following theorem.

Theorem 2: Class D(e) of psuedo cuts with respect to edge e can be expressed as

$$D(e) = \text{Min } \{x : x = \{e\} \oplus s,\ s \in S_s\} \tag{6}$$

Proof: Consider $\{e\} \oplus s'$ where s' is an edge disjoint union of cut sets $s_1, s_2, \ldots,$ and s_m. Then $\{e\} \oplus s'$ can be expressed as $\{e\} \oplus s_1 \oplus s_2 \oplus \ldots \oplus s_m$. If s' does not contain edge e, then $d \oplus s'$ is not in D(e) because $\{e\}$ is in D(e). Suppose edge e is in s'. Without loss of generality, let edge e be in s_1. Notice that s_1 is in S_s. Since $e \oplus s_1 \subset e \oplus s'$, set $\{e\} \oplus s'$ is not in D(e). Q.E.D.

By Theorem 2, we can use Equation (6) as the definition of D(e) which is obviously simpler to obtain D(e). However, we will see later that S can not be replaced by S_s when a class $D(e_1 e_2 \ldots e_m)$ of psuedo cuts with respect to edges $e_1, e_2, \ldots,$ and e_m for $m > 1$ is concerned.

There is a relationship between a psuedo cut in D(e) and class D(e) as follows.

Lemma 3: Let d be a psuedo cut in D(e). Then D(e) can be expressed as

$$D(e) = \text{Min } \{x : x = d \oplus s,\ s \in S\} \tag{7}$$

Proof: From Equation (3), we can see that any psuedo cut d in D(e) can be expressed as $\{e\} \oplus s'$ where $s' \in S$. Thus the right hand side of Equation (7) is equal to

$$\text{Min } \{x : x = \{e\} \oplus s' \oplus s,\ s \in S\} = \text{Min } \{x : x = \{e\} \oplus s'',\ s'' \in S\} \tag{8}$$

which is the definition of D(e). Notice that S is an Abelian group under $\oplus$. Q.E.D.

The use of Theorem 2 gives the relationship between classes D(e) and S_s as follows:

Lemma 4:

$$D(e) \otimes S_s = D(e) \qquad (9)$$

Proof: This lemma can be proven easily by knowing that (a) $\emptyset$ is in S_s, (b)

$$S_s \otimes S_s = S_s \qquad (10)$$

and (c) S in Equation (3) can be substituted by S_s by Theorem 2.

When G is nonseparable [2], D(e) has very interesting properties as

Theorem 3: Let G be nonseparable. Then

$$D(e) \otimes D(e) = S_s \qquad (11)$$

Before proving Theorem 3, we will prove the following lemma.

Lemma 5: If s_1 and s_2 are fundamental cut sets with respect to a tree in G [2], then $s_1 \oplus s_2$ is a cut set.

Proof: Let s_1 and s_2 be fundamental cut sets with respect to a tree T. Because each of s_1 and s_2 contains only one edge in T, deletion of all edges in s_1 and s_2 from G produces exactly three connected parts (an isolated vertex will be considered as one connected part). Let these parts be g_1, g_2, and g_3. For convenience, symbol

$\mathcal{E}(g_i, g_j)$ represents all edges each of which is connected between a vertex in g_i and a vertex in g_j. Without the loss of generality, let

$$s_1 = [\mathcal{E}(g_1,g_2)\ \mathcal{E}(g_1,g_3)] \tag{12}$$

and

$$s_2 = [\mathcal{E}(g_1,g_2)\ \mathcal{E}(g_2,g_3)] \tag{13}$$

Then $s_1 \oplus s_2$ will be

$$s_1 \oplus s_2 = \mathcal{E}(g_1,g_3)\ \mathcal{E}(g_2,g_3) \tag{14}$$

which is clearly a cut set in G. Q.E.D.

Proof of Theorem 3: From Theorem 1, it is clear that the ring sum of any two distinct psuedo cuts in D(e) is either a cut set or an edge disjoint union of cut sets. Hence, it is only necessary to show that for any cut set in G, there exists two psuedo cuts in D(e) such that the ring sum of these two psuedo cuts will give this cut set. Let s be a cut set.

Case 1: Suppose $e \in s$. Then $d = \{e\} \oplus s$ is in D(e). Also $\{e\}$ is in D(e). Hence $\{e\} \oplus d = s$.

Case 2: Suppose s does not contain edge e. Since G is non-separable, there exists a circuit C which contains edges e and e'_t where e'_t is in s. Let an edge sequence of the circuit C [2] be

$$e'_1\ e'_2\ \ldots\ e'_t\ \ldots\ e'_k,\ e,\ e_{k+1}\ \ldots\ e'_u\ \ldots\ e'_m$$

where e'_t and e'_u are in s and $e'_{t+1},\ldots,e'_k,\ e,\ e'_{k+1},\ldots,$ and e'_{u-1} are not in s as shown in Figure 2A.

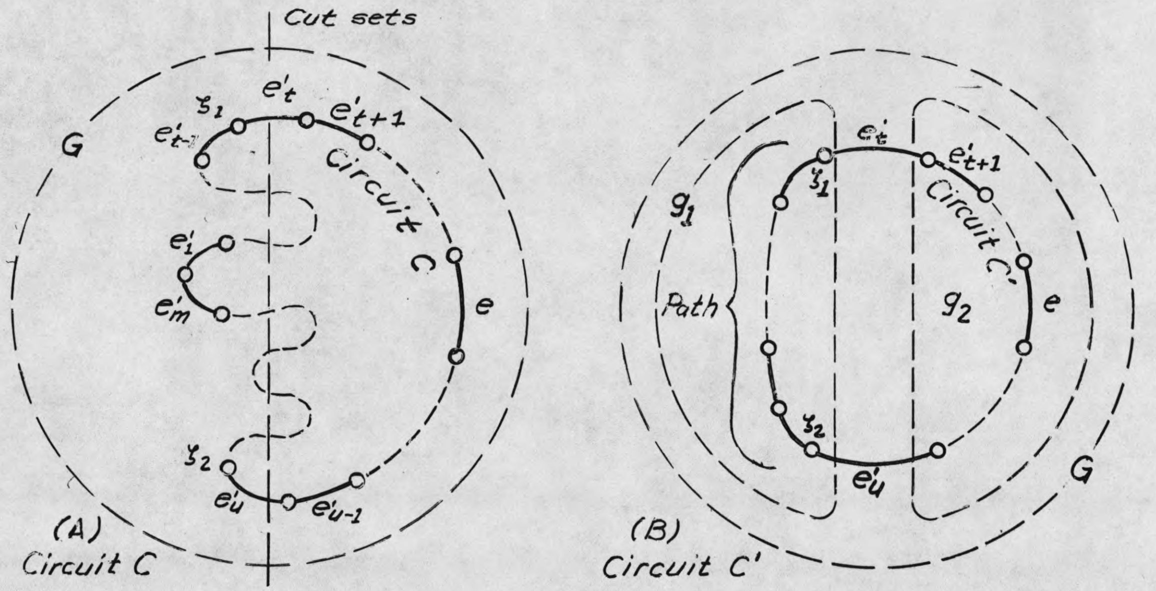

Fig. 2. Cut set s and edge e in G.

Let ζ_1 be the endpoint [2] of edges e'_{t-1} and e_t and ζ_2 be the endpoint of edges e'_u and e'_{u+1}. Also let g_1 and g_2 be the two connected parts obtained by deleting all edges in s where edge e is in g_2. Since g_1 is connected, there exists at least one path between ζ_1 and ζ_2 as shown in Figure 2B. Hence, there exists a circuit c' in G which contains edge e and exactly two edges in s. Notice that if ζ_1 and ζ_2 are the same, the existence of such a circuit is obvious. By choosing a tree T of G which contains all edges in the circuit c' except edge e'_t, we can obtain a fundamental cut set s' which contains edges e and e'_t. Notice that s is also a fundamental cut set with respect to T. Hence, by Lemma 5, s'', which is equal to $s \oplus s'$, is a cut set in G and edge e is in s''. Thus, there are two psuedo cuts $\{e\} \oplus s'$ and $\{e\} \oplus s''$ such that the ring sum of these two psuedo cuts gives the cut set s. Q.E.D.

Example 3: $D(e_0)$ in Example 1 is

$$D(e_0) = \Big\{\{e_0\},\ \{e_1e_3\},\ \{e_1e_4\},\ \{e_2e_3\},\ \{e_2e_4\}\Big\}$$

Thus

$$D(e_0) \otimes D(e_0) = \Big\{\emptyset, \{e_0e_1e_2\},\ \{e_0e_1e_4\},\ \{e_0e_2e_3\},\ \{e_0e_2e_4\},\ \{e_3e_4\}, \{e_1e_2\}\Big\}$$

which is equal to S_s. A common property of psuedo cuts in D(e) is given by the following theorem.

Theorem 4: Let $s \in S$ and $D(e) \neq S$. A set $\{e\} \oplus s$ is in D(e) if and only if $\{e\} \oplus s$ does not contain a cut set in S.

Proof: Suppose $\{e\} \oplus s$ contains a cut set s'. Then $\{e\} \oplus s \oplus s' \subset \{e\} \oplus s$. Thus $\{e\} \oplus s$ will not be in D(e). Suppose $\{e\} \oplus s$ is in D(e). Then s is a cut set but is not an edge disjoint union of cut sets by Theorem 2. Furthermore, edge e is in s, thus there is no cut set in $\{e\} \oplus s$. Q.E.D.

PSUEDO CUT WITH RESPECT TO SEVERAL EDGES

The previous section gives the definition and properties of psuedo cuts in D(e). Here we will generalize the definition of classes of psuedo cuts to include $D(e_1e_2 \ldots e_m)$.

Definition 6: A class $D(e_1 \ldots e_m)$ in G is defined as

$$D(e_1 \ldots e_m) = \text{Min}\ \{x : x = \{e_1 \ldots e_m\} \oplus s,\ s \in S\} \tag{15}$$

where edges $e_1, e_2, \ldots,$ and e_m are in G. When $D(e_1 \ldots e_m) \neq S_p$, a set in $D(e_1 \ldots e_m)$ is called a psuedo cut. $D(e_1 \ldots e_m)$ is a class of psuedo cuts with respect to edges $e_1, e_2, \ldots,$ and e_m.

Theorem 5: Let $D(e_1e_2) \neq S_s$, $D(e_1) \neq S_s$ and $D(e_2) \neq S_s$. Then

$$D(e_1e_2) = D(e_1) \otimes D(e_2) \tag{16}$$

Proof: Any set in $D(e_1) \otimes D(e_2)$ can be expressed as $\{e_1\} \oplus s_1 \oplus \{e_2\} \oplus s_2 = \{e_1e_2\} \oplus s$ where $s_1 \oplus s_2 = s$, s_1, s_2 and $s \in S$. Thus

$$D(e_1e_2) \supset D(e_1) \otimes D(e_2) \tag{17}$$

Suppose set $d = \{e_1e_2\} \oplus s$ is in $D(e_1e_2)$ but is not in $D(e_1) \otimes D(e_2)$. Let s be expressed as $s_1 \oplus s_2$ where $e_1 \in s_1$, and $s_1 \in S_s$. When $e_1 \in d$, s_1 is $\emptyset$ by definition. Then $\{e_1\} \oplus s_1 \in D(e_1)$. In order that d is not in $D(e_1) \otimes D(e_2)$, $\{e_2\} \oplus s_2$ can not be in $D(e_2)$. Because $\{e_2\} \oplus s_2$ is in $D(e_2)$ there must be a proper non-empty subset of $\{e_1e_2\} \oplus s$ in $D(e_1) \otimes D(e_2)$. However, if there is such a subset d' in $D(e_1) \otimes D(e_2)$, $d' \subset d$, d is not in $D(e_1e_2)$ which is a contradiction. Thus $\{e_2\} \oplus s_2$ can not be in $D(e_2)$. Then by Theorem 4, $\{e_2\} \oplus s_2$ must contain a cut set. Let this cut set be s'. Then $\{e_1e_2\} \oplus s \supset \{e_1e_2\} \oplus s \oplus s'$. Hence by Definitions 2 and 3, d is not in $D(e_1e_2)$ (contradiction). Thus

$$D(e_1e_2) \subset D(e_1) \otimes D(e_2) \tag{18}$$

Q.E.D.

From Theorem 5 we have

Lemma 6: Let e_1 and e_2 be in G. Then

$$\text{Min}\,\{\text{Min}\,y{:}y=\{e_1\}\oplus s,\ s \in S\} \otimes \text{Min}\,\{z{:}z=\{e_2\}\oplus s,\ s \in S\}\}$$

$$= \text{Min}\,\{\{y{:}y=\{e_1\}\oplus s,\ s \in S\} \otimes \{z{:}z=\{e_2\}\oplus s,\ s \in S\}\} \tag{19}$$

Similar to Theorem 4, we have the following.

Theorem 6: $\{e_1e_2\} \oplus s$ is in $D(e_1e_2)$ if and only if $\{e_1e_2\} \oplus s$ does not contain a cut set in S where $s \in S$.

Proof: Suppose $\{e_1e_2\} \oplus s$ contain a cut set s'. Then $\{e_1e_2\} \oplus (s \oplus s') = \{e_1e_2\} \oplus s'' \subset \{e_1e_2\} \oplus s$. Hence $\{e_1e_2\} \oplus s$ will not be in $D(e_1e_2)$. Suppose $\{e_1e_2\} \oplus s$ is in $D(e_1e_2)$, then there is no proper subset (non-empty) of $\{e_1e_2\} \oplus s$ in $D(e_1e_2)$. Thus $\{e_1e_2\} \oplus s$ can not contain a cut set in S. Q.E.D.

We will generalize Theorem 6 as follows.

Theorem 7: Let $s \in S$. A set $\{e_1 \ldots e_p\} \oplus s$ is in $D(e_1 \ldots e_p) \neq S_s$ if and only if $\{e_1 \ldots e_p\} \oplus s$ does not contain a cut set in S.

Proof: If $\{e_1 \ldots e_p\} \oplus s$ contains a cut set s', then $\{e_1 \ldots e_p\} \oplus s \oplus s' \subset \{e_1 \ldots e_p\} \oplus s$. Thus $\{e_1 \ldots e_p\} \oplus s$ will not be in $D(e_1 \ldots e_p)$. Suppose $\{e_1 \ldots e_p\} \oplus s$ is in $D(e_1 \ldots e_p)$. Then there is no non-empty proper subset of $\{e_1 \ldots e_p\} \oplus s$ is in $D(e_1 \ldots e_p)$. Thus it is impossible for $\{e_1 \ldots e_p\} \oplus s$ to contain a cut set. Q.E.D.

The relationship between $D(e_1 \ldots e_p)$ and $D(e_1)$, $D(e_2)$, ..., and $D(e_p)$ is given by the following.

Theorem 8: Let $D(e_1 \ldots e_p) \neq S_s$, $D(e_1 \ldots e_{p-1}) \neq S_s$ and $D(e_p) \neq S_s$ then

$$D(e_1 \ldots e_p) = D(e_1 \ldots e_{p-1}) \otimes D(e_p) \tag{20}$$

Proof: It is obvious that any set in $D(e_1 \ldots e_{p-1}) \otimes D(e_p)$ can be expressed as $\{e_1 \ldots e_p\} \oplus s$ where $s \in S$. Thus

$$D(e_1 \ldots e_p) \supset D(e_1 \ldots e_{p-1}) \otimes D(e_p) \tag{21}$$

Suppose set $d = \{e_1 \ldots e_p\} \otimes s \in D(e_1 \ldots e_p)$ is not in $D(e_1 \ldots e_{p-1}) \otimes D(e_p)$. Let s be

$$s = s_1 \oplus s_2 \oplus \ldots \oplus s_n \qquad (22)$$

where $s_1, s_2, \ldots, s_n \in S_s$ have no edge in common. Furthermore, let $e_p \in s_1$ (s_1 can be the empty set if $e_p \notin s$). Notice that s_i, $i=1,2,\ldots,n$ is a cut set if s_i is not the empty set. Then by Theorem 4, $\{e_p\} \oplus s_1 \in D(e_p)$. By Theorem 7, $\{e_1 \ldots e_{p-1}\} \oplus s_2 \oplus \ldots \oplus s_n$ is not in $D(e_1 \ldots e_{p-1})$ if $\{e_1 \ldots e_{p-1}\} \oplus s_2 \oplus \ldots \oplus s_n$ contains a cut set. Thus, in order that d is not in $D(e_1 \ldots e_{p-1}) \otimes D(e_p)$, $\{e_1 \ldots e_{p-1}\} \oplus s_2 \oplus \ldots \oplus s_n$ must contain a cut set s'. Then $\{e_1 \ldots e_p\} \oplus s \oplus s' \subset d$ is in $D(e_1 \ldots e_p)$ which contradicts the assumption that d is in $D(e_1 \ldots e_{p-1})$. Hence, $\{e_1 \ldots e_{p-1}\} \oplus s_2 \oplus \ldots \oplus s_n$ is in $D(e_1 \ldots e_{p-1})$. Thus, in order that d is in $D(e_1 \ldots e_p)$, d must be in $D(e_1 \ldots e_{p-1}) \otimes D(e_p)$. Q.E.D.

We can generalize the above theorem as follows.

Theorem 9: Let $D(e_1 \ldots e_p) \neq S_s$, $D(e_1 \ldots e_i) \neq S_s$ and $D(e_{i+1} \ldots e_p) \neq S_s$ where $1 \leq i < p$. Then

$$D(e_1 \ldots e_p) = D(e_1 \ldots e_i) \otimes D(e_{i+1} \ldots e_p) \qquad (23)$$

The proof is similar to that of Theorem 8.

Notice that the classes of psuedo cuts in the Theorems 8 and 9 must not be equal to S_s. In order to show that classes of psuedo cuts including S_s form an Abelian group under $\otimes$, we must remove such restrictions. It is necessary to study the following properties before discussing such a problem.

Lemma 7: Let d be in $D(e_1 \ldots e_p)$, then

$$D(e_1 \ldots e_p) = \text{Min } \{x : x = d \oplus s,\ s \in S\} \tag{24}$$

The proof is obvious from Definition 6.

Lemma 8: If $\{e_1 \ldots e_p\}$ is a cut set or an edge disjoint union of cut sets,

$$D(e_1 \ldots e_p) = S_s \tag{25}$$

Lemma 9: If $\{e_1 \ldots e_p\}$ contains a cut set $\{e_{i+1} \ldots e_p\}$ where $1 \leq i < p$, then

$$D(e_1 \ldots e_p) = D(e_1 \ldots e_i) \tag{26}$$

Proof: Since every set in $D(e_1 \ldots e_p)$ can be expressed as

$$\{e_1 \ldots e_p\} \oplus s = \{e_1 \ldots e_i\} \oplus \{e_{i+1} \ldots e_p\} \oplus s = \{e_1 \ldots e_i\} \oplus s' \tag{27}$$

where $\{e_{i+1} \ldots e_p\} \oplus s = s'$, the proof follows from Definition 6.

Lemma 10:

$$D(e_1 \ldots e_p) \otimes S_s = D(e_1 \ldots e_p) \tag{28}$$

The proof is clear because $\emptyset \in S_s$.

Theorem 3 gives that $D(e) \otimes D(e)$ is equal to S_s. The following is the generalization of Theorem 3.

Theorem 10: Let G be nonseparable. Then

$$D(e_1 \ldots e_p) \otimes D(e_1 \ldots e_p) = S_s \tag{29}$$

Proof: Case 1: When $D(e_1 \ldots e_p) = S_s$, then by Lemma 10, the left hand side of Equation (29) becomes

$$D(e_1 \ldots e_p) \otimes S_s = D(e_1 \ldots e_p) \tag{30}$$

which is equal to S_s. Case 2: When $D(e_1 \ldots e_p) \neq S_s$, then any set in $D(e_1 \ldots e_p) \otimes D(e_1 \ldots e_p)$ can be expressed as

$$\{e_1 \ldots e_p\} \oplus s_1 \oplus \{e_1 \ldots e_p\} \oplus s_2 = s_1 \oplus s_2 \tag{31}$$

which is in S_s. Hence

$$D(e_1 \ldots e_p) \otimes D(e_1 \ldots e_p) \subset S_s \tag{32}$$

Because of Lemma 9, we can assume that $(e_1 \ldots e_p)$ does not contain a cut set in S_s. Now we will prove that S_s is a subclass of $D(e_1 \ldots e_p) \otimes D(e_1 \ldots e_p)$. In the proof of Theorem 3, we show that for any set s in S_s, either e_1 is in s or there exists s_1 and s_2 in S_s such that $e_1 \in s_1$, $e_1 \in s_2$ and $s_1 \oplus s_2 = s$. Hence if $e_1 \in s$, $\{e_1\} \oplus s \in D(e_1)$ which gives that $\{e_1 \ldots e_p\} \oplus s$ is in $D(e_1 \ldots e_p)$. Also, we know by Theorem 7 that $\{e_1 \ldots e_p\} \in D(e_1 \ldots e_p)$. Thus, s is in $D(e_1 \ldots e_p) \otimes D(e_1 \ldots e_p)$. If $e_1 \notin s$, then $\{e_1\} \oplus s_1$ and $\{e_1\} \oplus s_2$ are in $D(e_1)$. Thus, $\{e_1 \ldots e_p\} \oplus s_1$ and $\{e_1 \ldots e_p\} \oplus s_2$ are in $D(e_1 \ldots e_p)$. Hence s is in $D(e_1 \ldots e_p) \otimes D(e_1 \ldots e_p)$. Thus

$$D(e_1 \ldots e_p) \otimes D(e_1 \ldots e_p) \supset S_s \tag{33}$$

Q.E.D.

Notice that G must be nonseparable. For example, if G is the graph in Figure 3, $D(e_1) = \left\{\{e_1\},\{e_2\}\right\}$ and $D(e_1) \otimes D(e_1) = \left\{\emptyset,\{e_1e_2\}\right\} \neq S_s$.

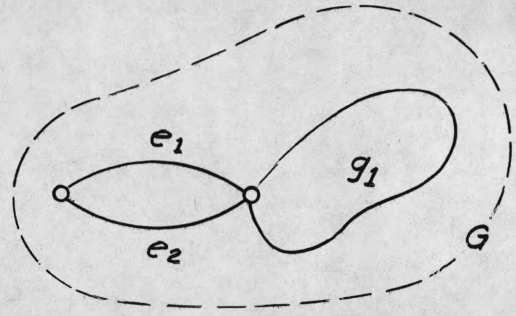

Fig. 3. A Separable Graph

Now we are ready to remove the restriction on classes of psuedo cuts in Theorem 9.

Theorem 11: Let G be nonseparable. Then

$$D(e_1 \ldots e_p) = D(e_1 \ldots e_i) \otimes D(e_{i+1} \ldots e_p) \tag{34}$$

where $1 \leq i < p$.

Proof: Suppose $D(e_1 \ldots e_p) = S_s$. Then if $D(e_{i+1} \ldots e_p) = S_s$, Equation (34) holds because of Lemmas 9 and 10. If both $D(e_1 \ldots e_i)$ and $D(e_{i+1} \ldots e_p)$ are not equal to S_s, Theorem 10 will give the proof. Thus, we only need to consider the case when $D(e_1 \ldots e_p) = S_s$. In this case, $\{e_1 \ldots e_p\}$ is either a cut set or an edge disjoint union of cut sets. When $D(e_1 \ldots e_i) = S_s$, Equation (34) holds because of Lemmas 9 and 10. Thus, we assume that $D(e_1 \ldots e_p) \neq S_s$ and $D(e_{i+1} \ldots e_p) \neq S_s$. Also by Lemma 9, we assume that $\{e_1 \ldots e_i\}$ and $\{e_{i+1} \ldots e_p\}$ do not contain a cut set. Since $\{e_1 \ldots e_p\} \in S$, either

$$\{e_1 \cdots e_i\} \oplus \{e_1 \cdots e_p\} = \{e_{i+1} \cdots e_p\} \tag{35}$$

or a proper nonzero subset of $\{e_{i+1} \cdots e_p\}$ is in $D(e_1 \ldots e_i)$. However, because $\{e_{i+1} \cdots e_p\}$ does not contain a cut set, $\{e_{i+1} \cdots e_p\}$ must be in $D(e_1 \ldots e_i)$ by Theorem 7. Thus, by Lemma 7,

$$D(e_1 \ldots e_i) = D(e_{i+1} \cdots e_p) \tag{36}$$

Hence, by Theorem 10, Equation (33) with $D(e_1 \ldots e_p) = S_s$ holds. Q.E.D.

The direct application of Theorem 11 gives the following important theorem.

Theorem 12: Let G be nonseparable. Then

$$[D(e_{11} \ldots e_{1m}) \otimes D(e_{21} \ldots e_{2m})] \otimes D(e_{31} \cdots e_{3p}) =$$

$$D(e_{11} \cdots e_{1m}) \otimes [D(e_{21} \ldots e_n) \otimes D(e_{31} \cdots e_{3p})] \tag{37}$$

Definition 7: Let $\mathcal{E}$ be a set of edges $e_1, e_2, \ldots,$ and e_p in G. Then $D(\mathcal{E})$ is a short hand notation of $D(e_1 e_2 \ldots e_p)$. Let $\mathcal{E} = \mathcal{E}_1 \odot \mathcal{E}_2$. Then $D(\mathcal{E}_1 \odot \mathcal{E}_2) = D(\mathcal{E})$ where $\odot$ is any operation defined in the set theory. With this notation, we have

Lemma 11: Let G be nonseparable. Then

$$D(\mathcal{E}_1) \otimes D(\mathcal{E}_2) = D(\mathcal{E}_1 \cup \mathcal{E}_2) = D(\mathcal{E}_1 \oplus \mathcal{E}_2) \tag{38}$$

The proof can be obtained by the use of Lemma 9 and Theorem 11. From Theorems 11 and 12, we know that

$$D(\mathcal{E}_1) \otimes D(\mathcal{E}_2) \otimes D(\mathcal{E}_3) = D(\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3) \tag{39}$$

where $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ are sets of edges in a nonseparable graph G. Hence, with Definition 6, we have the following theorem.

Theorem 13: Let G be nonseparable. Then

$$\text{Min}\left\{\text{Min}\ \{x:x=\mathcal{E}_1 \oplus s,\ s \in S\} \otimes \text{Min}\ \{y:y=\mathcal{E}_2 \oplus s,\ s \in S\} \otimes \right.$$

$$\left.\text{Min}\ \{z:z=\mathcal{E}_3 \oplus s,\ s \in S\}\right\} = \text{Min}\left\{\{x:x=\mathcal{E}_1 \oplus s,\ s \in S\} \otimes \right.$$

$$\left.\{y:y=\mathcal{E}_2 \oplus s,\ s \in S\} \otimes \{z:z=\mathcal{E}_3 \oplus s, s \in S\}\right\} \tag{40}$$

where $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ are sets of edges in G.

Definition 8: Collection $\{D\}$ is defined as the collection of all distinct classes of psuedo cuts in G and S_s. With this definition, it is easily seen that the previous results are enough to state that,

Theorem 14: Let G be nonseparable, then $\{D\}$ of G is an Abelian group under the operation $\otimes$, where S_s is a unit element and the inverse of $D(e_1 \ldots e_p)$ is itself.

Example 4: S_s of G in Figure 4 is

$$S_s = S = \left\{\emptyset,\ \{e_1e_2e_3\},\ \{e_1e_4e_6\},\ \{e_1e_2e_5e_6\},\ \{e_2e_3e_4e_6\},\right.$$

$$\left.\{e_3e_5e_6\},\ \{e_2e_4e_5\},\ \{e_1e_3e_4e_5\}\right\}$$

Then $D(e_1)$ is

$$D(e_1) = \text{Min}\ \{x:x=\{e_1\} \oplus s,\ s \in S_s\} = \{\{e_1\},\ \{e_2e_3\},\ \{e_4e_6\}, \{e_2e_5e_6\}, \{e_3e_4e_5\}\}$$

and $D(e_2)$ is

$$D(e_2) = \mathrm{Min}\{x: x=\{e_2\}\oplus s,\ s \in S_s\}=\{\{e_2\},\{e_1e_3\},\{e_1e_5e_6\},\{e_3e_4e_6\},\{e_4e_5\}\}$$

Thus, by Theorem 5 and Definition (3), $D(e_1e_2)$ is equal to

$$\mathrm{Min}\ \{x: x=d_1\oplus d_2,\ d_1 \in D(e_1),\ d_2 \in D(e_2)\} = D(e_1e_2) =$$

$$D(e_1)\otimes D(e_2) = \{\{e_1e_2\},\{e_3\},\{e_5e_6\},\{e_2e_4e_6\},\{e_1e_4e_5\}\}$$

On the other hand, $D(e_1e_2)$ is given by Definition 6 as

$$D(e_1e_2) = \mathrm{Min}\{x: x=\{e_1e_2\}\oplus s,\ s \in S\} = \{\{e_1e_2\},\{e_3\},\{e_2e_4e_6\},\{e_5e_6\},\{e_1e_4e_5\}\}$$

which is equal to $D(e_1) \otimes D(e_2)$. Notice that set $\{e_3\}$ is in $D(e_1e_2)$. Thus, by Lemma 7, $D(e_1e_2)$ can be expressed as

$$D(e_1e_2) = \mathrm{Min}\{x: x=\{e_3\}\oplus s,\ s \in S\} = \{\{e_3\},\{e_1e_2\},\{e_2e_4e_6\},\{e_5e_6\},\{e_1e_4e_5\}\}$$

which is, by Definition 6, equal to $D(e_3)$.

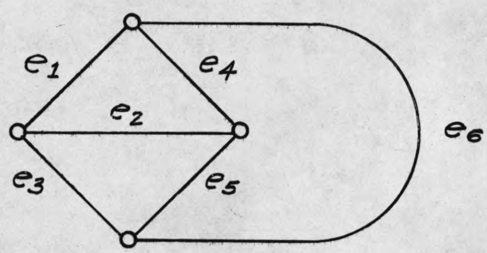

Fig. 4. A Linear Graph G

$[D(e_1) \otimes D(e_2)] \otimes D(e_3)$ is equal to $D(e_1e_2) \otimes D(e_3) = D(e_3) \otimes D(e_3) = S_s$. On the other hand, $D(e_1) \otimes [D(e_2) \otimes D(e_3)]$ is equal to $D(e_1) \otimes D(e_2e_3)$ where

$D(e_2e_3) = \mathrm{Min}\{x : x = \{e_2e_3\} \oplus s,\ s \in S\} = \{\{e_2e_3\}, \{e_1\}, \{e_4e_6\}, \{e_2e_5e_6\}, \{e_3e_4e_5\}\}$

Since $\{e_1\}$ is in $D(e_2e_3)$, by Lemma 7, $D(e_2e_3)$ is equal to $D(e_1)$. Thus,

$$D(e_1) \otimes D(e_2e_3) = D(e_1) \otimes D(e_1) = S_s.$$

Class $D(e_4)$ is given by

$$D(e_4) = \mathrm{Min}\{x : x = \{e_4\} \oplus s,\ s \in S\} = \{\{e_4\}, \{e_1e_6\}, \{e_2e_3e_6\}, \{e_2e_5\}, \{e_1e_3e_5\}\}$$

Consider $[D(e_1e_2) \otimes D(e_4)] \otimes D(e_3)$ where $D(e_1e_2) \otimes D(e_4)$ is

$$D(e_1e_2) \otimes D(e_4) = D(e_1e_2e_4) = \{\{e_1e_2e_4\}, \{e_3e_4\}, \{e_2e_6\}, \{e_4e_5e_6\}, \{e_1e_3e_6\}, \{e_1e_5\}, \{e_2e_3e_5\}\}$$

$D(e_1e_2e_4) \otimes D(e_3)$ will be

$$D(e_1e_2e_4) \otimes D(e_3) = \mathrm{Min}\{x : x = d_1 \oplus d_2,\ d_1 \in D(e_1e_2e_4),\ d_2 \in D(e_3)\} = \mathrm{Min}\{\{e_1e_2e_3e_4\}, \{e_4\}, \{e_2e_3e_6\}, \{e_3e_4e_5e_6\}, \{e_1e_6\}, \{e_1e_3e_5\}, \{e_2e_5\}, \{e_1e_2e_4e_5e_6\}\} = \{\{e_4\}, \{e_2e_3e_6\}, \{e_1e_6\}, \{e_1e_3e_5\}, \{e_2e_5\}\}$$

which is equal to $D(e_4)$. On the other hand

$$D(e_4) \otimes D(e_3) = \mathrm{Min}\{x : x = d_1 \oplus d_2,\ d_1 \in D(e_4),\ d_2 \in D(e_3)\} =$$

$$\{\{e_3e_4\},\{e_1e_2e_4\},\{e_1e_3e_6\},\{e_2e_6\},\{e_4e_5e_6\},\{e_2e_3e_5\},\{e_1e_5\}\} = D(e_3e_4)$$

Hence $D(e_1e_2) \otimes [D(e_4) \otimes D(e_3)]$ will be

$$D(e_1e_2) \otimes D(e_3e_4) = \mathrm{Min}\{x : x = d_1 \oplus d_2,\ d_1 \in D(e_1e_2),\ d_2 \in D(e_3e_4)\} =$$

$$\{\{e_4\},\{e_2e_3e_6\},\{e_1e_6\},\{e_1e_3e_5\},\{e_2e_5\}\}$$

which is equal to $D(e_4)$. Thus

$$[D(e_1e_2) \otimes D(e_4)] \otimes D(e_3) = D(e_1e_2) \otimes [D(e_4) \otimes D(e_3)].$$

PROPERTIES OF PSUEDO CUTS AND THEIR APPLICATION

Consider a connected linear graph G which consists of n_v vertices and n_e edges. Then the number of edges in a cut set can not exceed $n_e - n_v + 2$ because that any cut set can be a fundamental cut set of some tree in G and any fundamental cut set can not have more than the number of chords plus one edge. We will show here that a psuedo cut can not contain more than $n_e - n_v + 1$ edges and any psuedo cut is a subset of a set of chords in G.

From Definition 6 and Lemma 7, we know that if $D(e_{i_1} \ldots e_{i_m})$ and $D(e_{j_i} \ldots e_{j_n})$ have a psuedo cut in common then,

$$D(e_{i_1} \ldots e_{i_m}) = D(e_{j_1} \ldots e_{j_n}) \tag{41}$$

Conversely, we have the following lemma.

Lemma 12: Suppose set $\mathcal{E}$ of edges does not contain a cut set. Then for any subsets $\mathcal{E}_1$ and $\mathcal{E}_j$ in $\mathcal{E}$ where $\mathcal{E}_1 \neq \mathcal{E}_2$,

$$D(\mathcal{E}_1) \neq D(\mathcal{E}_2) \tag{42}$$

Proof: Suppose it is not true. Then $\mathcal{E}_1$ and $\mathcal{E}_2$ are in $D(\mathcal{E}_1)$. Thus, it is clear by Definition 6, $\mathcal{E}_1 \oplus \mathcal{E}_2$ is in S which contradicts the assumption that $\mathcal{E}$ does not contain a cut set. Q.E.D.

The maximum number of edges which a psuedo cut can have is given by the following theorem.

Theorem 15: The number of edges in a psuedo cut of a connected linear graph G can not exceed $n_e - n_v + 1$ where n_e is the number of edges and n_v is the number of vertices in G. Furthermore, there exists a psuedo cut which contains exactly $n_e - n_v + 1$ edges.

Proof: Suppose $\{e_1 \ldots e_m\}$ is a psuedo cut in $D(e_{i_1} \ldots e_{i_p})$. Then by Lemma 7,

$$D(e_{i_1} \ldots e_{i_p}) = \text{Min}\{x : x = \{e_1 \ldots e_m\} \oplus s,\ s \in S\} = D(e_1 \ldots e_m) \tag{43}$$

Also, by Lemma 10, $\{e_1 \ldots e_m\}$ contains no cut set in S. Thus, $m \leq n_e - n_v + 1$.

Let $\{e_1 \ldots e_c\}$ be a set of chords where $c = n_e - n_v + 1$. Then $\{e_1 \ldots e_c\}$ contains no cut sets. Thus, $\{e_1 \ldots e_c\}$ is a psuedo cut in $D(1_1 \ldots 1_c)$ by Lemma 10.

With these lemmas and theorem, we can see that (1) any set of chords in G belongs to exactly one class of psuedo cuts, and (2) any psuedo cut is a subset of a set of chords.

Definition 9: $\left\{D\{e_1 \ldots e_q\}\right\}$ is the subclass of $\{D\}$ of a linear graph G such that

$$\left\{D\{e_1\ldots e_q\}\right\} = \left\{D(e_{i_1}\ldots e_{i_p}):\{e_{i_1}\ldots e_{i_p}\}\subset\{e_1\ldots e_q\},\ p=1,2,\ldots,q\right\} \quad (44)$$

Notice that there are $2^q - 1$ classes in $\left\{D\{e_1\ldots e_q\}\right\}$. By Lemma 12, all classes in $\left\{D\{e_1\ldots e_q\}\right\}$ are distinct if $\{e_1\ldots e_q\}$ is a subset of a set of chords with respect to a tree in G. Suppose $\{e_1\ldots e_q\}$ does not contain a cut set. Also suppose G is connected. Consider a subgraph g of G which is obtained by deleting all edges in $\{e_1\ldots e_q\}$ from G. Since $\{e_1\ldots e_q\}$ does not contain a cut set, q is connected and contains all vertices of G. Thus, there exists a tree in g which is also a tree in G. Thus $\{e_1\ldots e_q\}$ is a subset of a set of chords with respect to a tree in G. Hence, we have the following Theorem.

Theorem 16: Let G be connected. Then there are $2^q - 1$ distinct classes in $\left\{D\{e_1\ldots e_q\}\right\}$ if and only if $\{e_1\ldots e_q\}$ is a subset of a set of chords with respect to a tree in G.

The following theorem is one of the important theorems in order to obtain $\{D\}$.

Theorem 17: Let $\{e_1,\ldots e_q\}$ be a set of chords in a nonseparable graph G. Then, any class of psuedo cuts of G is equal to one of the classes in $\left\{D\{e_1\ldots e_q\}\right\}$.

Proof: Let T be a tree corresponding to set $\{e_1\ldots e_q\}$ of chords. Consider a class $D(e_{j_1}\ldots e_{j_u})$. It is obvious that when $\{e_{j_1}\ldots e_{j_u}\}\subset\{e_1\ldots e_q\}$, $D(e_{j_1}\ldots e_{j_u}) \in \left\{D\{e_1\ldots e_q\}\right\}$. Thus, we assume that $\{e_{j_1}\ldots e_{j_u}\}\not\subset\{e_1\ldots e_q\}$. Because of Lemma 8, we can further assume that $\{e_{j_1}\ldots e_{j_u}\}$ does not contain a cut set. Without the loss of generality, let $\{e_{j_1} \ldots e_{j_u}\} - \{e_1 \ldots e_q\} = \{e_{j_1} \ldots e_{j_m}\}$ where $1 < m \leq u$. Notice that edges $e_{j_1} \ldots e_{j_m}$ are in T. Also, let $s_1, \ldots, s_m$ be fundamental cut sets of $e_{j_1}, e_{j_2}, \ldots, e_{j_m}$ with respect to T such that $e_{j_p} \epsilon$ s_p for $p=1,2,\ldots,m$. Then, by Theorem 11, $D(e_{j_1} \ldots e_{j_u})$ can be expressed as

$$D(e_{j_1} \ldots e_{j_m}) = D(e_{j_1}) \otimes D(e_{j_2}) \otimes \ldots \otimes D(e_{j_m}) \otimes D(e_{j_{m+1}} \ldots e_{j_u}) \quad (45)$$

Furthermore, by Theorem 11

$$D(e_{j_p}) = D(s_p - \{e_{j_p}\}) \quad (46)$$

we have

$$D(e_{j_1} \ldots e_{j_m}) = D(s_1 - \{e_{j_1}\}) \otimes D(s_2 - \{e_{j_2}\}) \otimes \ldots \otimes$$

$$D(s_m - \{e_{j_m}\}) \otimes D(e_{j_{m+1}} \ldots e_{j_u}) = D(s_1 - \{e_{j_1}\} \otimes$$

$$\left\{s_2 - \{e_{j_2}\}\right\} \oplus \ldots \oplus \left\{s_m - \{e_{j_m}\} \oplus \{e_{j_{m+1}} \ldots e_{j_u}\}\right\}) \quad (47)$$

Notice that every edge in $s_p - \{e_{j_p}\}$ is in the set $\{e_1 \ldots e_q\}$ of chords with respect to T. Hence

$$\left\{s_1 - \{e_{j_1}\}\right\} \oplus \ldots \oplus \left\{s_m - \{e_{j_m}\}\right\} \oplus \{e_{j_{m+1}} \ldots e_{j_u}\} \subset \{e_1 \ldots e_q\} \quad (48)$$

Thus $D(e_{j_1} \ldots e_{j_m}) \epsilon \left\{D\{e_1 \ldots e_q\}\right\}$ Q.E.D.

This theorem gives that $\{D\} = \left\{\{D(e_1 \ldots e_q)\}, S_s\right\}$ if $\{e_1 \ldots e_q\}$ is a set of chords in a nonseparable graph G. Thus, in order to obtain all possible distinct classes of psuedo cuts, we only need to obtain $\left\{D\{e_1 \ldots e_q\}\right\}$ where $\{e_1 \ldots e_q\}$ is a set of chords.

There are so-called topological formulas which analyze a linear electrical network which requires knowing all possible sets of chords in G corresponding to the electrical network. The properties of psuedo cuts discussed here yield to obtain all possible such sets by knowing one set of chords and S of G. Also, by Lemma 7, we know that any set of chords is in exactly one class of psuedo cut in $\left\{D\{e_1 \ldots e_q\}\right\}$ where $\{e_1 \ldots e_q\}$ is a set of chords. Hence, it is not necessary to test for duplications of sets of chords. The following example will clarify this point.

Example 5: All possible sets of chords in G shown in Figure 5 can be obtained by the following procedure. Take a tree, say $\{acd\}$. Then we can obtain a set of fundamental cut sets with respect to $\{acd\}$ as $\{abf\}$, $\{bce\}$, and $\{def\}$. Then

$$S = \left\{\emptyset, \{abf\}, \{bce\}, \{def\}, \{acef\}, \{abde\}, \{bcdf\}, \{acd\}\right\}$$

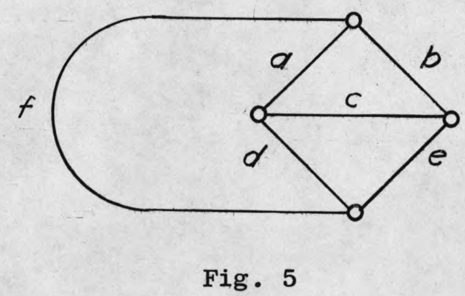

Fig. 5

The set of chords corresponding to $\{acd\}$ is $\{bed\}$. Thus $\{D\{bed\}\}$ is the collection of all distinct classes of psuedo cuts which consists of the following.

$$D(b) = \{\{b\},\{af\},\{ce\},\{ade\},\{cdf\}\}$$

$$D(e) = \{\{e\},\{bc\},\{df\},\{acf\},\{abd\}\}$$

$$D(f) = \{\{f\},\{ab\},\{de\},\{ace\},\{bcd\}\}$$

$$D(be) = \{\{be\},\{c\},\{bdf\},\{ad\},\{aef\}\}$$

$$D(bf) = \{\{bf\},\{a\},\{bde\},\{cd\},\{cef\}\}$$

$$D(ef) = \{\{ef\},\{abe\},\{bcf\},\{d\},\{ac\}\}$$

and

$$D(bef) = \{\{bef\},\{ab\},\{cf\},\{bd\},\{adf\}\}$$

Thus, all possible sets of chords are these psuedo cuts which consist of $n_e - n_v + 1 = 3$, that is,

$$\{ade\} \text{ and } \{cdf\} \in D(b)$$

$$\{acb\} \text{ and } \{abd\} \in D(e)$$

$$\{ace\} \text{ and } \{bcd\} \in D(f)$$

$$\{bdf\} \text{ and } \{aef\} \in D(be)$$

$$\{bde\} \text{ and } \{cef\} \in D(bf)$$

$$\{abe\} \text{ and } \{bef\} \in D(ef)$$

$$\{bef\} \text{ and } \{adf\} \in D(bef)$$

are all possible sets of chords in G.

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